Pumping Lemma

How to use it?

(some heuristic rules)

1. Consider the given language, what is the condition you should break so that the string wouldn't belong to the language. Can be e.g.

   - About numbers of some characters in the string (are dependent), e.g. $(a^k b^m c^m,\ a^m b^{2m})$
   - About prime numbers, e.g. $(a^p,\ p\ some\ prime)$
   - About parts of world: the first and later part of the word are dependent, e.g. $(ww^{\mathcal{R}},\ ww)$

2. Think what is the simplest string, which contains the members of condition. There can be irrelevant parts like in $a^k b^m c^m$ the number of a's doesn't matter anything (select e.g. $b^m c^m$).

   - Sometimes you need one character to separate the members of condition, e.g. in $a^m b^k a^m$ you should be able to distinguish first a's and later a's (select e.g. $a^m b a^m$).
   - If the first part and latter part of the word depend on each other, we should distinguish first and later, but otherwise the parts can be anything (e.g. in language ww $(w \in \{a,b\}^*)$ we can select $a^m b a^m b$ or $b a^m b a^m$).

3. Select now that mystical n such that the first member of condition belongs to first n characters of the string and can be pumped. The other goal (which is not crucial, but saves work) is to select such n that number of possible divisions of x into parts uvw is small.

   ⋆ e.g. $x = a^m b^m$. Select $n = m$, when parts uv belong to a^m (w may contain only b's or first a's and then b's) we can pump a's and break the condition.

   NB! We can also choose $n = 2m$, which causes more work. Now the pumped part may contain only a's, only b's on both (of form $a^i b^j$). In the last case 2-time pumping breaks conditions ($a^i b^j a^i b^j$ doesn't belong to the language).

4. Now test **all** possible divisions by the rules $x = uvw$, $|uv| \leq n$ ja $v \neq \epsilon$. For each division test pumping by $i = 0, 2, 3,\ldots$ until you find such i that $uv^iw \notin A$. Usually it is enough to try $i = 0$ or $i = 2$ to get the wanted result.

5. If you managed to break the condition for all divisions uvw, you can shout: Heureka! Congratulations!

Why we use it in this way?

PL says:

$$R(A) \quad \Rightarrow \quad \exists n \, \forall x \; P(x) \; \exists \; uvw \; Q(x, u, v, w) \; \forall i \; S(x).$$

Contraposition:

$$A \Rightarrow B \quad \equiv \quad \neg B \Rightarrow \neg A.$$

Also holds:

$$\begin{aligned} \neg \exists x P(x) &= \forall x \neg P(x) \\ \neg \forall x P(x) &= \exists x \neg P(x). \end{aligned}$$

We can turn the PL around and still holds

$$\begin{aligned} & \neg(\exists n \; \forall x \; P(x) \; \exists uvw \; Q(x, u, v, w) \; \forall i \; S(x, i)) && \Rightarrow \quad \neg R(A). \\ \equiv \quad & \forall n \; \neg(\forall x \; P(x) \; \exists uvw \; Q(x, u, v, w) \; \forall i \; S(x, i)) && \Rightarrow \quad \neg R(A) \\ \equiv \quad & \forall n \; \exists x \; P(x) \; \neg(\exists uvw \; Q(x, u, v, w)) \; \forall i \; S(x, i)) && \Rightarrow \quad \neg R(A) \\ \equiv \quad & \forall n \; \exists x \; P(x) \; \forall uvw \; Q(x, u, v, w) \; \neg(\forall i S(x, i)) && \Rightarrow \quad \neg R(A) \\ \equiv \quad & \forall n \; \exists x \; P(x) \; \forall uvw \; Q(x, u, v, w) \; \exists i \; \neg(S(x, i)) && \Rightarrow \quad \neg R(A). \end{aligned}$$

So: if for all n there is some x $|x| \geq n$, for all uvw $|uv| \leq n$, $|v| \geq 1$ there is some $i \geq 1$ such that $uv^iw \notin A$, then A is not regular.