Exercise session 1

1. Pigeonhole Principle says: If you have more pigeons than pigeonholes, and each pigeon flies into some pigeonhole, then there must be at least one hole that has more than one pigeon.

   What happens, if you have as many pigeonholes as there are natural numbers, and as many pigeons, as there are integers? What about, if you have as many pigeons as there are natural numbers, but each pigeon tries to make nest with every other pigeon into a different hole? (Only one nest can be made into one hole.)

2. There is coming an infinite number of busses, with infinite number of people in each of them. How would you allocate the guests into rooms of Hilbert's Hotel?

3. In the quest book of Hilbert's Hotel there is only finite number of names in each page and new quests must always write their names into the next empty line. How many pages there must be in the book so that there is room for the new names (without reorganizing the names) as long as there is room in the hotel (maybe after reorganizing)?

4. There lives a huge amount of inhabitants in the Cantor's planet – in fact as many as there are real numbers (i.e. $|\mathbb{R}|$). Also the names of people are very strange: each name is a real number, which is represented with infinite precision (and each has a unique name). Inhabitants 0.0000..... and 1.0000.... have decided to arrange a group tour to Hilbert's Hotel, and they have invited all inhabitants whose names are in order between their own names (i.e. in interval $]0,1[$). Can you accommodate everybody into Hilbert's Hotel? Justify your answer carefully! (Hint: Cantor's diagonalization method.)

5. Find the error in the following proof that $2 = 1$. Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a - b)(a + b) = b(a - b)$, and divide each side by $(a-b)$, to get $a+b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.

6. Let X be a set and X the size of $n = |X|$. Prove by induction that the size of the powerset of X is $|\mathcal{P}(X)| = 2^n$.

7. What is wrong in the following induction proof that all cats are of the same colour?

   Let n be the number of cats. If $n = 1$ the claim holds clearly (one cat is always of the same colour). Let's now suppose that for any group of n cats the claim holds. Then let's consider a group of $n + 1$ cats. By selecting any n cats from this group (which can be done in $n + 1$ different ways) we get by the induction assumption a group in which all the cats have the same colour. So all $n + 1$ cats must be of the same colour.

8. Prove the following claim. If there are $n(n \geq 2)$ people in the party, then at least two people have equal number of friends in the party.

9. Prove by contraposition: If c is an odd integer number, then the equation $n^2 + n - c = 0$ doesn't have any integer solution for n.